Eigenvalues of a Special Tridiagonal Matrix

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Abstract

In this paper we consider a special tridiagonal test matrix. We prove that its eigenvalues are the even integers $2, \ldots, 2n$ and show its relationship with the famous Kac-Sylvester tridiagonal matrix.

1 Introduction

We begin with a quick overview of the theory of symmetric tridiagonal matrices, that is, we detail a few basic facts about tridiagonal matrices. In particular, we describe the symmetrization process of a tridiagonal matrix as well as the orthogonal polynomials that arise from the characteristic polynomials of said matrices.

Definition 1.1. *A **tridiagonal matrix**, T_n, is of the form:*

$$T_n = \begin{bmatrix} a_1 & b_1 & 0 & \ldots & 0 \\ c_1 & a_2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a_{n-1} & b_{n-1} \\ 0 & \ldots & 0 & c_{n-1} & a_n \end{bmatrix}, \tag{1.1}$$

where entries below the subdiagonal and above the superdiagonal are zero. If $b_i \neq 0$ for $i = 1, \ldots, n-1$ and $c_i \neq 0$ for $i = 1, \ldots, n-1$, T_n is called a *Jacobi* matrix. In this paper we will use a more compact notation and only describe the subdiagonal, diagonal, and superdiagonal (where appropriate). For example, T_n can be rewritten as:

$$T_n = \begin{pmatrix} & b_1 & & \ldots & & b_{n-1} & \\ a_1 & & a_2 & \ldots & a_{n-1} & & a_n \\ & c_1 & & \ldots & & c_{n-1} & \end{pmatrix}. \tag{1.2}$$

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Note that the study of symmetric tridiagonal matrices is sufficient for our purpose as any Jacobi matrix with $b_i c_i > 0 \; \forall i$ can be symmetrized through a similarity transformation:

$$A_n = D_n^{-1} T_n D_n = \begin{pmatrix} & \sqrt{b_1 c_1} & & \cdots & & \sqrt{b_{n-1}c_{n-1}} & \\ a_1 & & a_2 & \cdots & a_{n-1} & & a_n \\ & \sqrt{b_1 c_1} & & \cdots & & \sqrt{b_{n-1}c_{n-1}} & \end{pmatrix}, \tag{1.3}$$

where $D_n = \mathrm{diag}(\gamma_1, \ldots, \gamma_{\mathrm{n}})$ and $\gamma_i = \sqrt{\dfrac{c_i c_{i+1} \cdots c_{n-1}}{b_i b_{i+1} \cdots b_{n-1}}}$.
We refer the reader to [1] for a proof and more detailed exposition. The added symmetry allows for an easier analysis of the spectrum of A_n. In particular, a cofactor expansion along the last row of $P_n = A_n - \lambda I_n$ yields the recurrence relations:

$$P_0(\lambda) = 1 \tag{1.4}$$
$$P_1(\lambda) = a_1 - \lambda \tag{1.5}$$
$$P_i(\lambda) = (a_i - \lambda)P_{i-1}(\lambda) - b_{i-1}c_{i-1}P_{i-2}(\lambda). \tag{1.6}$$

Here $\{P_i\}$ is an orthogonal family of polynomials with respect to the inner product:

$$\langle P_n, P_m \rangle := \int_{-\infty}^{+\infty} P_n(x) P_m(x) w(x) dx, \tag{1.7}$$

where $w(x)$ is the measure or weight function $w(x) = e^{-x^2}$. Orthogonality yields the following useful properties (see [9]) :

$$\text{The zeros of } P_i \text{ are real,} \qquad 1 \le i \le n, \tag{1.8}$$
$$\text{The zeros of } P_i \text{ and } P_{i+1} \text{ interlace,} \qquad 1 \le i \le n-1. \tag{1.9}$$

In other words, the eigenvalues of A_n are real and the eigenvalues of A_{i-1} interlace those of A_i for $1 \le i \le n$. An interesting problem in matrix theory is that of the *inverse eigenvalue problem* (IEP). Before formally stating the problem for tridiagonal matrices, let us introduce some notation.

Definition 1.2. *Given T_n an $n \times n$ tridiagonal matrix, the $(n-1) \times (n-1)$* ***principal submatrix****, $\hat{T}_n$, is the matrix formed by removing the last row and column of T_n.*

IEP for Tridiagonal Matrices. *Given the ordered lists $\Lambda = (\lambda_i)_{i=1}^n$ and $\Theta = (\theta_i)_{i=1}^{n-1}$ such that Θ interlaces Λ, i.e., $\lambda_i \le \theta_i \le \lambda_{i+1}$ for $i = 1, \ldots, n$, find the $(n \times n)$ symmetric tridiagonal matrix T_n such that Λ and Θ are the spectra of T_n and $\hat{T}_n$, respectively.*

Note that the existence and uniqueness (up to signs) of T_n from spectral data is only guaranteed when Λ and Θ strictly interlace; we refer the reader to [7] for more details. Also, the *IEP* for tridiagonal matrices is fully solved in the sense that given the lists Λ and Θ, one can reconstruct T_n algorithmically (see [8], [5, page 473]).

In this paper, we are interested in the tridiagonal *test matrix* W_n that has spectrum $\Lambda = \{2, 4, \ldots, 2n\}$ and $\hat{W}_n$ has spectrum $\Theta = \{3, 5, \ldots, 2n-1\}$. By *test matrix* we mean a matrix with known eigenvalues and given structure. Such matrices make it possible to test the stability of numerical eigenvalue algorithms. The motivation behind W_n is provided in *section 2*.

A famous tridiagonal matrix is the *Kac-Sylvester* matrix proposed by Clement [2] as a test matrix.

Definition 1.3. *The* $(n+1)\times(n+1)$ ***Kac-Sylvester matrix***, K_n, *is:*

$$K_n = \begin{pmatrix} & n & n-1 & \dots & 2 & 1 & \\ 0 & 0 & & \dots \quad \dots & & 0 & 0 \\ & 1 & 2 & \dots & n-1 & n & \end{pmatrix}. \tag{1.10}$$

It has the particularly nice eigenvalues : $\sigma(K_n) = \{2k-n\}_{k=0}^{n}$. There are several proofs that K_n has the above spectrum (see [3], [6], [10]). The relevance of this matrix will become apparent when we prove our main result.

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2 Motivation

The problem of finding the tridiagonal matrix T_n with the spectrum of T_n being $\Lambda = \{2, 4, \dots, 2n\}$ and the spectrum of $\hat{T_n}$ being $\Theta = \{3, 5, \dots, 2n-1\}$ was posed by one of my research supervisors, Dr. N. B. Willms. It arises from the study of spring-mass systems in free motion, where the eigenvalues correspond to natural frequencies of the systems. It turns out that many spring-mass systems beget tridiagonal matrices (see [4]), where the entries of the corresponding tridiagonal matrix are functions of the spring constants and masses of the systems.

More specifically, given n masses $\{m_i\}_{i=1}^{n}$ in suspension from a ceiling with the respective spring constants $\{k_i\}_{i=0}^{n-1}$, where the hanging end of the system is free (called a *fixed-free* system), we wish to model this system in terms of matrices. The solutions of $|\lambda M - EKE^{-1}| = 0$ are precisely the natural frequencies of the system (see [4, page 45]), where M, K, and E are given by:

$$M = \operatorname{diag}(m_1, m_2, \dots, m_{n-1}, m_n). \tag{2.1}$$

$$K = \begin{pmatrix} & -k_1 & & \dots & & -k_{n-1} & \\ k_0+k_1 & & k_1+k_2 & \dots & k_{n-2}+k_{n-1} & & k_{n-1} \\ & -k_1 & & \dots & & -k_{n-1} & \end{pmatrix}, \tag{2.2}$$

$$E = \begin{bmatrix} 1 & -1 & 0 & \dots & \dots & 0 \\ 0 & 1 & -1 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 & 0 \\ \vdots & \ddots & \ddots & 0 & 1 & -1 \\ 0 & \dots & \dots & 0 & 0 & 1 \end{bmatrix} \quad \text{and } E^{-1} = \begin{bmatrix} 1 & 1 & \dots & \dots & 1 & 1 \\ 0 & 1 & 1 & \ddots & \ddots & 1 \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 & \vdots \\ \vdots & \dots & \dots & 0 & 1 & 1 \\ 0 & \dots & \dots & 0 & 0 & 1 \end{bmatrix}, \tag{2.3}$$

Note that E is upper-bidiagonal and that E^{-1} is upper-triangular with all ones. As such our problem is to verify the form of the symmetric tridiagonal matrix $B = L^{-1}EKE^{-1}L^{-T}$ (with $L = \operatorname{diag}(\sqrt{m_1}, \sqrt{m_2}, \dots, \sqrt{m_n})$) with eigenvalues Λ and such that $\sigma(\hat{B}) = \Theta$. As mentioned earlier, there exists an algorithm for the *IEP* for tridiagonal matrices. Our problems differs from the former as we wish to find T_n with entries as explicit functions of n.

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3 Main Result

We begin with a definition of the matrix of interest which we shall show to be the solution of the *IEP*.

Definition 3.1. *Let $W_n(k)$ be the $n \times n$ symmetric tridiagonal matrix with the following entries:*

$$W_n(k) = \begin{cases} a_i = k, & i = 1, \ldots, n \\ b_i = \sqrt{\frac{i(2n-1-i)}{4}}, & i = 1, \ldots, n-2 \\ b_{n-1} = \frac{\sqrt{n(n-1)}}{\sqrt{2}}, & \end{cases}$$

as per *definition* (1.1). For example,

$$W_6(7) = \begin{bmatrix} 7 & \sqrt{\frac{5}{2}} & 0 & 0 & 0 & 0 \\ \sqrt{\frac{5}{2}} & 7 & \sqrt{\frac{9}{2}} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{9}{2}} & 7 & \sqrt{\frac{12}{2}} & 0 & 0 \\ 0 & 0 & \sqrt{\frac{12}{2}} & 7 & \sqrt{\frac{14}{2}} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{14}{2}} & 7 & \sqrt{\frac{30}{2}} \\ 0 & 0 & 0 & 0 & \sqrt{\frac{30}{2}} & 7 \end{bmatrix}.$$

We are now ready to introduce and prove our main result:

Theorem 3.2. *The spectra of $W_n(n+1)$ and $\hat{W}_n(n+1)$ are $\Lambda = \{2, 4, \ldots, 2n\}$ and $\Theta = \{3, 5, \ldots, 2n-1\}$, respectively.*

Proof. Part A - Eigenvalues of $W_n(n+1)$
By the *Schur* decomposition theorem, there exists unitary $Q \in M_{n\times n}(\mathbb{C})$ such that $Q^{-1}W_n(n+1)Q = U$ is upper-triangular with the eigenvalues of $W_n(n+1)$ being on the diagonal of U. Hence, subtracting $(n+1)I_n$ from $W_n(n+1)$, where I_n is the $n \times n$ identity matrix, shifts the eigenvalues by $-(n+1)$. Note that $W_n(n+1) - (n+1)I_n = W_n(0)$, the $(n \times n)$ symmetric tridiagonal matrix with zero diagonal.
Now, choose $H = \mathrm{diag}(\gamma_1, \ldots, \gamma_n)$ such that,

$$\tilde{W}_n(0) := 2HW_n(0)H^{-1} = \begin{pmatrix} & 1 & & 2 & & \ldots & & n-2 & & 2(n-1) & \\ 0 & & 0 & & \ldots & & \ldots & & 0 & & 0 \\ & 2n-2 & & 2n-3 & & \ldots & & n+1 & & n & \end{pmatrix}. \tag{3.1}$$

That is, given that the $i^{th}, (i+1)^{th}$ (as per *definition* (1.1)), $i = 1, \ldots, n-2$, entry of $W_n(0)$ is b_i, the corresponding entry of $HW_n(0)H^{-1}$ on the superdiagonal is $\frac{\gamma_i}{\gamma_{i+1}}b_i$. We find γ_i,γ_{i+1} such that $\frac{\gamma_i}{\gamma_{i+1}}b_i = \frac{i}{2}$ for $i = 1, \ldots, n-2$. Note that b_{n-1} introduces an extra factor of 2 so that $\frac{\gamma_{n-1}}{\gamma_n}b_{n-1}$ is twice as large as the pattern suggests. In other words $\frac{\gamma_{n-1}}{\gamma_n}b_{n-1} = n-1$. On the other hand, for the subdiagonal, since $\frac{\gamma_i}{\gamma_{i+1}}b_i = \frac{i}{2}$, we have that $\frac{\gamma_{i+1}}{\gamma_i} = \frac{2}{i}b_i$ and thus $\frac{\gamma_{i+1}}{\gamma_i}b_i = \frac{2}{i}b_i^2$. Simplifying, $\frac{2}{i}b_i^2 = \frac{2}{i}(\frac{2ni-i-i^2}{4}) = \frac{2n-1-i}{2}$ with $i = 1, \ldots, n-2$. Finally, knowing that $\frac{\gamma_{n-1}}{\gamma_n}b_{n-1} = n-1$, we can determine the corresponding subdiagonal entry. Specifically, $\frac{\gamma_n}{\gamma_{n-1}} = \frac{1}{n-1}b_{n-1}$, so that, $\frac{\gamma_n}{\gamma_{n-1}}b_{n-1} = \frac{1}{n-1}b_{n-1}^2 = \frac{1}{n-1}(\frac{n(n-1)}{2}) = \frac{n}{2}$. Our matrix is now precisely as claimed in (3.1) save for the extra factor of 2 which we have removed to simplify later calculations. In other words, the magnitude of the eigenvalues of $\tilde{W}_n(0)$ are twice those of $W_n(0)$. Now, consider $J(\tilde{W}_n(0))^T J$ where J is the exchange matrix defined with the Kronecker-Delta where $(J)_{ij} = \delta_{i,n-j+1}$. The matrix, $J(\tilde{W}_n(0))^T J$, is of the form:

$$X := J(\tilde{W}_n(0))^T J = \begin{pmatrix} & 2(n-1) & n-2 & \dots & & 2 & 1 \\ 0 & & 0 & \dots & \dots & 0 & 0 \\ & n & n+1 & \dots & 2n-3 & 2n-2 & \end{pmatrix}. \tag{3.2}$$

In the spirit of [6]'s proof that K_n (*definition 1.3*) has the spectrum $\sigma(K_n) = \{2k-n\}_{k=0}^{n}$, let us introduce the $n \times n$ matrix P_1 such that:

$$P_1 = \begin{bmatrix} 1 & 0 & \dots & \dots & \dots & 0 \\ -1 & 1 & 0 & \dots & \dots & 0 \\ 0 & -1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \dots & 0 & -1 & 1 & 0 \\ 0 & \dots & \dots & 0 & -1 & 1 \end{bmatrix} \quad \text{where} \quad P_1^{-1} = \begin{bmatrix} 1 & 0 & \dots & \dots & \dots & 0 \\ 1 & 1 & 0 & \dots & \dots & 0 \\ 1 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \dots & \dots & 1 & 1 & 0 \\ 1 & \dots & \dots & 1 & 1 & 1 \end{bmatrix} \tag{3.3}$$

so that P_1 is lower-bidiagonal and P_1^{-1} is lower-triangular with all ones. We claim that,

$$\mathring{W}_n(0) := P_1 X P_1^{-1} = \left[\begin{array}{c|c} 2(n-1) & \mathbf{v} \\ \hline 0 & \\ \vdots & L_{n-1} - I_{n-1} \\ 0 & \end{array}\right], \tag{3.4}$$

where $\mathbf{v} = [2(n-1) \quad \underbrace{0 \quad \dots \quad 0}_{(n-2) zeros}]$ and,

$$L_{n-1} = \begin{pmatrix} & n-2 & n-3 & \dots & & 2 & 1 \\ -(n-1) & & 0 & \dots & \dots & 0 & 0 \\ & n & n+1 & \dots & 2n-4 & 2n-3 & \end{pmatrix}. \tag{3.5}$$

We prove the above claim. The entries of X are (as row vectors):

$$\begin{aligned} \text{First row} &= [0 \quad 2(n-1) \quad 0 \quad \dots \quad 0], \\ i^{th} \text{ row} &= [0 \quad \dots \quad 0 \quad \underbrace{n+i-2}_{(\text{i}-1)^{\text{th}} \text{ entry}} \quad 0 \quad \underbrace{n-i}_{(\text{i}+1)^{\text{th}} \text{ entry}} \quad 0 \quad \dots \quad 0], \\ \text{Last row} &= [0 \quad \dots \quad 0 \quad 2n-2 \quad 0]. \end{aligned}$$

The i^{th} row of $P_1 X$ is the $(i-1)^{th}$ row of X subtracted from the i^{th} row of X. Hence, the first row is left unchanged. Therefore, the rows of $P_1 X$ are:

$$\begin{aligned} \text{First row} &= [0 \quad 2(n-1) \quad 0 \quad \dots \quad 0], \\ i^{th} \text{ row} &= [0 \quad \dots \quad 0 \quad \underbrace{-(n+i-3)}_{(\text{i}-2)^{\text{th}} \text{ entry}} \quad \underbrace{n+i-2}_{(\text{i}-1)^{\text{th}} \text{ entry}} \quad \underbrace{-(n+i+1)}_{\text{i}^{\text{th}} \text{ entry}} \quad \underbrace{n-i}_{(\text{i}+1)^{\text{th}} \text{ entry}} \quad 0 \quad \dots \quad 0], \\ \text{Last row} &= [0 \quad \dots \quad 0 \quad 2n-2 \quad 0]. \end{aligned}$$

Now, postmultiplying by P_1^{-1} adds columns together in the sense that $(P_1XP_1^{-1})_{i,j}$, $j = 1, \dots, n$ (for each row i) is obtained by adding the $(n-j+1)^{th}$ last columns of $(P_1X)_i$. In this manner, the first row of $P_1XP_1^{-1}$ is:

$$[2(n-1) \quad 2(n-1) \quad 0 \quad \dots \quad 0].$$

The second row of P_1X is:

$$[n \quad -2(n-1) \quad n-2 \quad 0 \quad \dots \quad 0],$$

so that the second row of $P_1XP_1^{-1}$ is:

$$[n-2(n-1)+(n-2) \quad -2(n-1)+(n-2) \quad n-2 \quad 0 \quad \dots \quad 0].$$

Or, more simply,

$$[0 \quad -n \quad n-2 \quad 0 \quad \dots \quad 0].$$

Now, the entries of the i^{th} row, $i = 3, \dots, n-1$, of $P_1XP_1^{-1}$ are:

$$\begin{aligned}
-(n+i-3)+(n+i-2)-(n-i+1)+(n-i) &= 0, && (\mathrm{i}-2)^{\mathrm{th}} \text{ entry},\\
(n+i-2)-(n-i+1)+(n-i) &= n+i-3, && (\mathrm{i}-1)^{\mathrm{th}} \text{ entry},\\
-(n-i+1)+(n-i) &= -1, && \mathrm{i}^{\mathrm{th}} \text{ entry},\\
(n-i) &= n-i, && (\mathrm{i}+1)^{\mathrm{th}} \text{ entry}.
\end{aligned}$$

More succinctly, the i^{th} row of $P_1XP_1^{-1}$ is:

$$[0 \quad \dots \quad 0 \quad \underbrace{n+i-3}_{(\mathrm{i}-1)^{\mathrm{th}} \text{ entry}} \quad \underbrace{-1}_{\mathrm{i}^{\mathrm{th}} \text{ entry}} \quad \underbrace{n-i}_{(\mathrm{i}+1)^{\mathrm{th}} \text{ entry}} \quad 0 \quad \dots \quad 0].$$

Finally, the last row, $i = n$, is,

$$[0 \quad \dots \quad 0 \quad 2n-3 \quad -1].$$

From where we get that $\mathring{W}_n(0) = P_1XP_1^{-1}$ looks like:

$$\mathring{W}_n(0) = \left[\begin{array}{c|ccccc}
2(n-1) & 2(n-1) & 0 & \dots & \dots & 0 \\ \hline
0 & -n & n-2 & 0 & \dots & 0 \\
\vdots & n & -1 & n-3 & \ddots & \vdots \\
\vdots & \ddots & n+1 & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & 1 \\
0 & 0 & \dots & 0 & 2n-3 & -1
\end{array}\right], \tag{3.6}$$

which is precisely as claimed in (3.4). Now, since $\mathring{W}_n(0)$ is block upper-triangular, we have that $\sigma(\mathring{W}_n(0)) = \{2(n-1)\} \cup \sigma(L_{n-1} - I_{n-1})$. Let $\overset{+}{L}_{n-1}$ be the matrix obtained from taking the absolute value of all the entries of L_{n-1}, that is,

$$\overset{+}{L}_{n-1} = \begin{pmatrix}
 & n-2 & n-3 & \dots & 2 & 1 \\
n-1 & 0 & \dots & \dots & 0 & 0 \\
 & n & n+1 & \dots & 2n-4 & 2n-3
\end{pmatrix}. \tag{3.7}$$

Note that $\sigma(\overset{+}{L}_{n-1}) = -\sigma(L_{n-1})$, that is, their eigenvalues have opposite signs. To see why, consider,

$$D_{n-1} = \operatorname{diag}(1, -1, \dots, (-1)^{\mathrm{n}}). \tag{3.8}$$

Clearly $D_{n-1}^{-1} = D_{n-1}$. Then,

$$D_{n-1}L_{n-1}D_{n-1}^{-1} = \begin{pmatrix} & -(n-2) & -(n-3) & \dots & -2 & -1 & \\ -(n-1) & 0 & \dots & \dots & 0 & 0 \\ & -n & -(n+1) & \dots & -(2n-4) & -(2n-3) & \end{pmatrix}. \tag{3.9}$$

So, $\sigma(L_{n-1}) = \sigma(D_{n-1}L_{n-1}D_{n-1}^{-1})$. Then, $-(D_{n-1}L_{n-1}D_{n-1}^{-1}) = \overset{+}{L}_{n-1}$, but multiplying $D_{n-1}L_{n-1}D_{n-1}^{-1}$ by -1 changes the signs of all of its eigenvalues. And so we have that $\sigma(\overset{+}{L}_{n-1}) = (-1)\sigma(L_{n-1})$. Additionally, $\overset{+}{L}_{n-1}$ shares the same characteristic polynomial as the following matrices:

$$(\overset{+}{L}_{n-1})^T = \begin{pmatrix} & n & n+1 & \dots & 2n-4 & 2n-3 & \\ n-1 & 0 & \dots & \dots & 0 & 0 \\ & n-2 & n-3 & \dots & 2 & 1 & \end{pmatrix}, \tag{3.10}$$

$$J(\overset{+}{L}_{n-1})^T J = \begin{pmatrix} & 1 & 2 & \dots & n-3 & n-2 & \\ 0 & 0 & \dots & \dots & 0 & n-1 \\ & 2n-3 & 2n-4 & \dots & n+1 & n & \end{pmatrix}, \tag{3.11}$$

since they all share the same characteristic polynomial. From where, consider the $(2n-2)\times(2n-2)$ block matrix:

$$B = \left[\begin{array}{c|c} J(\overset{+}{L}_{n-1})^T J & 0 \\ \hline 0 & (\overset{+}{L}_{n-1})^T \end{array}\right] \tag{3.12}$$

$$= \begin{pmatrix} & 1 & 2 & \dots & n-2 & 0 & n & \dots & 2n-4 & 2n-3 & \\ 0 & \dots & \dots & 0 & n-1 & n-1 & 0 & \dots & \dots & 0 \\ & 2n-3 & 2n-4 & \dots & n & 0 & n-2 & \dots & 2 & 1 & \end{pmatrix}.$$

Furthermore, consider the $(2n-2)\times(2n-2)$ *perfect shuffle* matrix:

$$S_{n-1} = \left[\begin{array}{c|c|c} e_{n-1} & 0 & o_{n-1} \\ \hline 0 & S_{n-2} & 0 \\ \hline o_{n-1} & 0 & e_{n-1} \end{array}\right], \tag{3.13}$$

where $o_{n-1} = \frac{1-(-1)^{n-1}}{2}$ and $e_{n-1} = \frac{1+(-1)^{n-1}}{2}$. For example,

$$S_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.14}$$

Note that S_{n-1} has orthogonal columns and so $S_{n-1}S_{n-1}^T = I_{2n-2}$. However, S_{n-1} is also symmetric, so $S_{n-1}^2 = I_{2n-2}$, from where we have that $S_{n-1} = S_{n-1}^{-1}$. Now,

$$S_{n-1}BS_{n-1} = \left[\begin{array}{c|c|c} e_{n-1} & 0 & o_{n-1} \\ \hline 0 & S_{n-2} & 0 \\ \hline o_{n-1} & 0 & e_{n-1} \end{array}\right] \left[\begin{array}{c|ccc|c} 0 & 1 & \dots & 0 & 0 \\ \hline 2n-3 & & & & 0 \\ 0 & & & & \vdots \\ \vdots & & B' & & 0 \\ 0 & & & & 2n-3 \\ \hline 0 & \dots & 0 & 1 & 0 \end{array}\right] \left[\begin{array}{c|c|c} e_{n-1} & 0 & o_{n-1} \\ \hline 0 & S_{n-2} & 0 \\ \hline o_{n-1} & 0 & e_{n-1} \end{array}\right], \tag{3.15}$$

where B' is obtained by deleting the first and last rows and columns of B. Premultiplying by S_{n-1} will swap the first and last rows of B, followed by a postmultiplication by S_{n-1} which swaps the first and last column, in this fashion we obtain,

$$S_{n-1}B = \left[\begin{array}{c|ccccc|c} 0 & e_{n-1} & 0 & \dots & 0 & o_{n-1} & 0 \\ \hline (2n-3)S_{n-2}\mathbf{e_1} & & & S_{n-2}B' & & & (2n-3)S_{n-2}\mathbf{e_{2n-4}} \\ \hline 0 & o_{n-1} & 0 & \dots & 0 & e_{n-1} & 0 \end{array}\right]. \tag{3.16}$$

So,

$$S_{n-1}BS_{n-1} = \left[\begin{array}{c|c|c} 0 & ZS_{n-2} & 0 \\ \hline (2n-3)S_{n-2}Z^T & S_{n-2}B'S_{n-2} & (2n-3)S_{n-2}Q^T \\ \hline 0 & QS_{n-2} & 0 \end{array}\right], \tag{3.17}$$

where $Z = [e_{n-1} \quad 0 \quad \dots \quad 0 \quad o_{n-1}]$, $Q = [o_{n-1} \quad 0 \quad \dots \quad 0 \quad e_{n-1}]$, and $\mathbf{e_i}$ are the standard basis vectors. Consider the topmost middle block of $S_{n-1}BS_{n-1}$, in other words, $[e_{n-1} \quad 0 \quad \dots \quad 0 \quad o_{n-1}]S_{n-2}$. If $(n-1)$ is even, then Z picks off the first row of S_{n-2} so that $(n-2)$ is odd, so that the first row of S_{n-2} is $[0 \quad \dots \quad 0 \quad 1]$. On the other hand, if $(n-1)$ is odd, then Z picks off the last row of S_{n-2} when $(n-2)$ is even, so that the last row of S_{n-2} is $[0 \quad \dots \quad 0 \quad 1]$.
A similar argument applied to the remaining three outer-middle blocks and given that the two innermost elements of B, $n-1$ and $n-1$, are already in their proper place, the following form for $S_{n-1}BS_{n-1}$ follows by induction,

$$G_{2n-2} = S_{n-1}BS_{n-1} = \begin{bmatrix} 0 & \dots & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 2 & 0 & 2n-3 \\ \vdots & ⋰ & ⋰ & ⋰ & 2n-4 & 0 \\ 0 & 2n-4 & ⋰ & ⋰ & ⋰ & \vdots \\ 2n-3 & ⋰ & 2 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & \dots & 0 \end{bmatrix}. \tag{3.18}$$

G_{2n-2} is tridiagonal with zero antidiagonal; it also bears close resemblance to the *Kac-Sylvester* matrix previously mentioned in *definition 1.3*, specifically, $G_{2n-2} = {K_{2n-3}}^T J$. Since L resembles the *Kac-Sylvester* matrix, consider the following $(2n-2)\times(2n-2)$ matrix,

$$P_2(n) = \begin{bmatrix} 1 & 0 & \dots & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & \dots & 0 \\ -1 & 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 & 0 \\ \vdots & \dots & \ddots & 0 & 1 & 0 \\ 0 & \dots & 0 & -1 & 0 & 1 \end{bmatrix} \quad \text{where} \quad P_2^{-1}(n) = \begin{bmatrix} 1 & 0 & \dots & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & \dots & 0 \\ 1 & 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \dots & \dots & 0 & 1 & 0 \\ o_n & \dots & 0 & 1 & 0 & 1 \end{bmatrix}, \tag{3.19}$$

and $P_2(n)$ only has one off-diagonal band of negative ones. Note that $P_2^{-1}(n)$ is banded lower-triangular with diagonals of alternating zeros or ones and that $o_n = \frac{1-(-1)^{n-1}}{2}$. For example,

$$P_2^{-1}(4) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}. \tag{3.20}$$

Now, rows $i = 2, \dots, 2n-3$ of G_{2n-2} are given by:

$$[0 \quad \dots \quad 0 \quad \underbrace{i}_{\text{(i}-1)^{\text{th}}\text{ entry}} \quad 0 \quad \underbrace{(2n-3)-i+2}_{\text{(i+1)}^{\text{th}}\text{ entry}} \quad 0 \quad \dots \quad 0].$$

Premultiplying by $P_2^{-1}(n)$ adds rows $(i-k)$, $k_{odd} < i$ to row i, $i = 3, \dots, 2n-2$. In other words, for the i^{th} column of G_{2n-2}:

$$[0 \quad \dots \quad 0 \quad \underbrace{i}_{\text{(i}-1)^{\text{th}}\text{ entry}} \quad 0 \quad \underbrace{(2n-3)-i}_{\text{(i+1)}^{\text{th}}\text{ entry}} \quad 0 \quad \dots \quad 0]^T,$$

the corresponding column of $[P_2^{-1}(n)]G_{2n-2}$ is:

$$[0 \quad \dots \quad 0 \quad \underbrace{i}_{\text{(i}-1)^{\text{th}}\text{ entry}} \quad 0 \quad \underbrace{(2n-3)}_{\text{(i+1)}^{\text{th}}\text{ entry}} \quad 0 \quad \underbrace{(2n-3)}_{\text{(i+3)}^{\text{th}}\text{ entry}} \quad \dots \quad (2n-3) \text{ or } 0]^T.$$

With 0 occuring when the size of the matrix, n, is odd. Now, $[P_2^{-1}(n)]G_{2n-2}$ has a checkerboard pattern with entries $(2n-3)$ below the anti-diagonal. For example,

$$[P_2^{-1}(4)]G_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 5 \\ 0 & 0 & 3 & 0 & 5 & 0 \\ 0 & 4 & 0 & 5 & 0 & 5 \\ 5 & 0 & 5 & 0 & 5 & 0 \\ 0 & 5 & 0 & 5 & 0 & 5 \end{bmatrix}.$$

Now, the i^{th} row of $[P_2^{-1}(n)]G_{2n-2}$ is:

$$[0 \quad \dots \quad 0 \quad \underbrace{i}_{\text{(i}-1)^{\text{th}}\text{ entry}} \quad 0 \quad \underbrace{(2n-3)}_{\text{(i+1)}^{\text{th}}\text{ entry}} \quad \underbrace{0 \quad (2n-3) \quad 0 \quad \dots \quad 0 \text{ or } (2n-3)}_{\text{n-i+1 terms}}],$$

with $(2n-3)$ and 0 alternating across the row and $(2n-3)$ occurring at the end of the row if $n+i-1$ is even. Now, postmultiplying by $P_2(n)$ subtracts the $(i+2)^{th}$ column from the i^{th} one, leaving the last two columns unchanged. Since $[P_2^{-1}(n)]G_{2n-2}$ is checkerboard below the anti-diagonal, every $(2n-3)$ entry not in the last two columns is deleted. More specifically, the i^{th} row of $[P_2^{-1}(n)]G_{2n-2}P_2(n)$ is now:

$$[0 \quad \dots \quad 0 \quad \underbrace{-i}_{\text{(i−3)}^{\text{th}}\text{ entry}} \quad 0 \quad \underbrace{i-(2n-3)}_{\text{(i−1)}^{\text{th}}\text{ entry}} \quad 0 \quad \underbrace{(2n-3)-(2n-3)}_{\text{(i+1)}^{\text{th}}\text{ entry}} \quad 0 \quad \dots \quad 0 \quad x \quad y],$$

for $i = 2, \dots, 2n-5$ and where $x = (2n-3) - y$ and $y = \frac{(2n-3)(1-(-1)^{i+1})}{2}$. Equivalently, the i^{th} row is:

$$[0 \quad \dots \quad 0 \quad \underbrace{-i}_{\text{(i−3)}^{\text{th}}\text{ entry}} \quad 0 \quad \underbrace{-((2n-5)-i+2)}_{\text{(i−1)}^{\text{th}}\text{ entry}} \quad 0 \quad \dots \quad 0 \quad x \quad y].$$

Note that the first row is:

$$[0 \quad \dots \quad 0 \quad -1 \quad 0 \quad 1 \quad 0].$$

Since the last two rows of $[P_2^{-1}(n)]G_{2n-2}$ are:

$$\begin{bmatrix} (2n-3) & 0 & (2n-3) & 0 & \dots & 0 & y & x \\ 0 & (2n-3) & 0 & (2n-3) & \dots & 0 & x & y \end{bmatrix},$$

and given that postmultiplication by $P_2(n)$ subtracts the $(i+2)^{th}$ column from the i^{th} one leaving the last two unchanged, we get that $[P_2^{-1}(n)]G_{2n-2}[P_2(n)]$ has its last two rows as:

$$\begin{bmatrix} 0 & 0 & \dots & 0 & y & x \\ 0 & 0 & \dots & 0 & x & y \end{bmatrix}.$$

And hence we get that,

$$P_2^{-1}(n)G_{2n-2}P_2(n) = \left[\begin{array}{c|c} -G_{2n-4} & V_{(2n-4)\times 2} \\ \hline 0_{2\times(2n-4)} & F_{2\times 2} \end{array}\right], \tag{3.21}$$

where $F = \begin{bmatrix} 2n-3 & 0 \\ 0 & 2n-3 \end{bmatrix}$ since G_{2n-2} is always of even size.

We then have that the eigenvalues of (3.21) are $\{2n-3, 2n-3\} \cup \sigma(-G_{2n-4})$. Iterating, we get that,

$$\sigma(G_{2n-2}) = \{(2n-3), (2n-3), -(2n-5), -(2n-5), \dots, (-1)^n, (-1)^n\}. \tag{3.22}$$

Therefore B constructed from $J(\overset{+}{L}_{n-1})^T J$ and $(\overset{+}{L}_{n-1})^T$ will have each eigenvalue of even multiplicity. Also, $\sigma(G_{2n-2}) = \sigma(B)$ by a similarity transformation, thus,

$$\sigma(J(\overset{+}{L}_{n-1})^T J) = \sigma(\overset{+}{L}_{n-1}) = \{(2n-3), -(2n-5), (2n-7), \dots, (-1)^n\}. \tag{3.23}$$

Since $\sigma(\overset{+}{L}_{n-1}) = -\sigma(L_{n-1})$,

$$\sigma(L_{n-1}) = \{-(2n-3), (2n-5), -(2n-7), \dots, (-1)^{n-1}\}, \tag{3.24}$$

so,

$$\sigma(L_{n-1} - I_{n-1}) = \{-(2n-2), (2n-6), -(2n-6), \dots, r\}, \tag{3.25}$$

with $r = (-1)^n - 1$. Therefore, we have,

$$\sigma(\mathring{W}_n(0)) = \{2(n-1)\} \cup \{-(2n-2), (2n-6), -(2n-6), \ldots, r\}. \tag{3.26}$$

Since $\mathring{W}_n(0)$ and $\tilde{W}_n(0)$ are similar (see (3.2)), we get,

$$\sigma(\frac{1}{2}\tilde{W}_n(0)) = \{(n-1)\} \cup \{-(n-1), (n-3), -(n-3), \ldots, \frac{1}{2}r\}. \tag{3.27}$$

It is worth tidying up (3.27). Doing so, we obtain,

$$\sigma\left(\frac{1}{2}\tilde{W}_n(0)\right) = \sigma(W_n(0)) = \{2k-(n+1)\}_{k=1}^{n}, \tag{3.28}$$

whereby,

$$\sigma(W_n(n+1)) = \{2k\}_{k=1}^{n} = \{2, 4, \ldots, 2n\}. \tag{3.29}$$

Part B - Eigenvalues of $\hat{W}_n(n+1)$

We wish to show that $\sigma(\hat{W}_n(n+1)) = \{3, 5, \ldots, 2n-1\}$. Note that the entries of $\sigma(\hat{W}_n(n+1))$ lack the special term b_{n-1}, and so we can consider the principal submatrix of (3.1). In other words,

$$M := 2\hat{H}\hat{W}_n(0)\hat{H}^{-1} = \begin{pmatrix} & 1 & & 2 & & \ldots & & n-3 & & n-2 & \\ 0 & & 0 & & \ldots & & \ldots & & 0 & & 0 \\ & 2n-2 & & 2n-3 & & \ldots & & n+2 & & n+1 & \end{pmatrix}. \tag{3.30}$$

Note that M is of size $(n-1)\times(n-1)$. Now, recall the matrix P_1, (3.3), now of size $(n-1)\times(n-1)$, and consider the product $P_1^{-1}MP_1$. Premultiplying by P_1^{-1} adds rows k, $k = 1, \ldots, i-1$, to the i^{th} row leaving the first row unchanged. We have that the i^{th} row, $i = 2, \ldots, n-2$, of M is:

$$[0 \quad \ldots \quad 0 \quad \underbrace{(2n-i)}_{\text{(i−1)}^{\text{th}}\text{ entry}} \quad 0 \quad \underbrace{i}_{\text{(i+1)}^{\text{th}}\text{ entry}} \quad 0 \quad \ldots \quad 0].$$

The last row is:

$$[0 \quad \ldots \quad 0 \quad n+1 \quad 0].$$

It is easier to consider the i^{th} column, $i = 2, \ldots, n-2$, for which the entries are:

$$[0 \quad \ldots \quad 0 \quad \underbrace{i-1}_{\text{(i−1)}^{\text{th}}\text{ entry}} \quad 0 \quad \underbrace{(2n-i-1)}_{\text{(i+1)}^{\text{th}}\text{ entry}} \quad 0 \quad \ldots \quad 0]^T,$$

so that the sum of the entries of the column is $(i-1)+(2n-i-1) = 2n-2$. Since premultiplying by P_1^{-1} adds rows k, $k = 1, \ldots, i-1$, to the i^{th} row, anything below the main diagonal becomes $(2n-2)$ while anything on the diagonal has the entry directly above it (from the above row) added to it. The i^{th} row of $P_1^{-1}M$ is now:

$$[2n-2 \quad \ldots \quad 2n-2 \quad \underbrace{(2n-i)+(i-2)}_{\text{(i−1)}^{\text{th}}\text{ entry}} \quad \underbrace{(i-1)}_{\text{i}^{\text{th}}\text{ entry}} \quad \underbrace{i}_{\text{(i+1)}^{\text{th}}\text{ entry}} \quad 0 \quad \ldots \quad 0],$$

or,

$$[\underbrace{2n-2 \quad \ldots \quad 2n-2}_{i-2\text{ terms}} \quad \underbrace{2n-2}_{\text{(i−1)}^{\text{th}}\text{ entry}} \quad \underbrace{(i-1)}_{\text{i}^{\text{th}}\text{ entry}} \quad \underbrace{i}_{\text{(i+1)}^{\text{th}}\text{ entry}} \quad 0 \quad \ldots \quad 0]. \tag{3.31}$$

Now, postmultiplying by P_1 subtracts the $(i+1)^{th}$ column from the i^{th} one.
In this manner, (3.31) becomes:

$$[0 \quad \dots \quad 0 \quad \underbrace{(2n-2)-(i-1)}_{\text{(i−1)}^{\text{th}}\text{ entry}} \quad \underbrace{(i-1)-i}_{\text{i}^{\text{th}}\text{ entry}} \quad \underbrace{i}_{\text{(i+1)}^{\text{th}}\text{ entry}} \quad 0 \quad \dots \quad 0].$$

More simply put:

$$[0 \quad \dots \quad 0 \quad (2n-i-1) \quad -1 \quad i \quad 0 \quad \dots \quad 0],$$

and where $i = 2, \dots, n-2$. Note that the last row of $P_1^{-1}M$ is:

$$[\underbrace{2n-2 \quad \dots \quad 2n-2}_{n-2 \text{ terms}} \quad n-2],$$

so that the last row of $P_1^{-1}MP_1$ is:

$$[0 \quad \dots \quad 0 \quad \underbrace{(2n-2-(n-2))}_{\text{(n−2)}^{\text{th}}\text{ entry}} \quad (n-2)],$$

or,

$$[0 \quad \dots \quad 0 \quad \underbrace{n}_{\text{(n−2)}^{\text{th}}\text{ entry}} \quad (n-2)].$$

We finally get that $P_1^{-1}MP_1$ is:

$$P_1^{-1}MP_1 = \frac{1}{2}\begin{pmatrix} & 1 & & 2 & & \dots & & n-3 & & n-2 & \\ -1 & & -1 & & \dots & & \dots & & -1 & & n-2 \\ & 2n-3 & & 2n-4 & & \dots & & n+1 & & n & \end{pmatrix} = J(\overset{+}{L}_{n-1})^T J - I_{n-1}. \tag{3.32}$$

But we know from (3.24) that,

$$\sigma(L_{n-1}) = \{-(2n-3), (2n-5), \dots, (-1)^{n-1}\}. \tag{3.33}$$

So, by (3.9) and (3.10), we have:

$$\sigma(J(\overset{+}{L}_{n-1})^T J) = \{(2n-3), -(2n-5), \dots, (-1)^n\}. \tag{3.34}$$

And therefore,

$$\sigma(J(\overset{+}{L}_{n-1})^T J - I_{n-1}) = \{(2n-4), -(2n-4), \dots, 0 \text{ or } -2\}, \tag{3.35}$$

with -2 occurring when n is odd. From where we have,

$$\sigma(M) = \{(n-2), -(n-2), (n-4), -(n-4) \dots, 0 \text{ or } -1\}. \tag{3.36}$$

More simply,

$$\sigma(\hat{W}_n(0)) = \{2k-n\}_{k=1}^{n-1}. \tag{3.37}$$

Finally, we indeed have that,

$$\sigma(\hat{W}_n(n+1)) = \{2k+1\}_{k=1}^{n-1}. \tag{3.38}$$

□

As a final result, it is worth noting that $W_n(n-1)$ has a particularly nice Cholesky decomposition.

Theorem 3.3. $W_n(n-1) = LL^T$ *where*

$$\begin{cases} L_{i,i} = \sqrt{\frac{2}{i}} b_i & \forall i = 1, \ldots, n-2 \\ L_{i+1,i} = \sqrt{\frac{i}{2}} & \forall i = 1, \ldots, n-2 \\ L_{n-1,n-1} = \sqrt{\frac{1}{n-1}} b_{n-1} & \\ L_{n,n-1} = n-1 & \\ L_{n,n} = 0 & \end{cases}$$

Proof. We must show that,

$$LL^T = \begin{pmatrix} & l_1\alpha_1 & & l_2\alpha_2 & \ldots & l_{n-2}\alpha_{n-2} & & l_{n-1}\alpha_{n-1} & \\ l_1^2 & & \alpha_1^2 + l_2^2 & & \ldots & \ldots & \alpha_{n-2}^2 + l_{n-1}^2 & & \alpha_{n-1}^2 + l_n^2 \\ & l_1\alpha_1 & & l_2\alpha_2 & \ldots & l_{n-2}\alpha_{n-2} & & l_{n-1}\alpha_{n-1} & \end{pmatrix} = W_n(n-1), \tag{3.39}$$

where,

$$L = \begin{pmatrix} & 0 & 0 & \ldots & 0 & & 0 & \\ l_1 & & l_2 & \ldots & \ldots & l_{n-1} & & l_n \\ & \alpha_1 & \alpha_2 & \ldots & \alpha_{n-2} & & \alpha_{n-1} & \end{pmatrix}. \tag{3.40}$$

In other words, we must verify the following equalities,

$$\begin{cases} l_1^2 = n-1 & \\ l_i\alpha_i = \sqrt{\frac{2ni-i-i^2}{4}} & \forall i = 1, \ldots, n-2 \\ \alpha_i^2 + l_{i+1}^2 = n-1 & \forall i = 1, \ldots, n-1 \\ l_{n-1}\alpha_{n-1} = \sqrt{\frac{2ni-i-i^2}{2}} & \end{cases} \tag{3.41}$$

Now, for the first case,

$$\begin{aligned} l_1^2 &= 2 \cdot b_1^2 \\ &= 2 \cdot \frac{2n-2}{4} \\ &= n-1. \end{aligned}$$

For the second case,

$$\begin{aligned} l_i\alpha_i &= \sqrt{\frac{2}{i}} \cdot b_i \cdot \sqrt{\frac{i}{2}} \\ &= \sqrt{\frac{2ni-i-i^2}{4}}. \end{aligned}$$

And for the third case,

$$\begin{aligned}\alpha_i^2 + l_{i+1}^2 &= \frac{i}{2} + \frac{2}{i+1} b_{i+1}^2 \\ &= \frac{i(i+1) + 4 \cdot b_{i+1}^2}{2(i+1)} \\ &= \frac{i(i+1) - (i+1)(i-2n+2)}{2(i+1)} \\ &= n-1.\end{aligned}$$

Finally,

$$\begin{aligned}\alpha_{n-1} l_{n-1} &= \sqrt{\frac{1}{n-1}} \cdot b_{n-1} \cdot \sqrt{n-1} \\ &= \sqrt{\frac{2ni - i - i^2}{2}}.\end{aligned}$$

From where we have that $LL^T = W_n(n-1)$. □

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